Atmospheric Refraction and Dispersion

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Introduction

Let us introduce the **optical length** traveled by a light ray across a material of constant and isotropic refraction index n as:

$$S = n \cdot L, \tag{1}$$

with L the (physical) length traveled by the ray. Since the traveled length is just the speed of light times the time, we get also:

$$S = n \cdot L = n \cdot vt \tag{2}$$

As the speed of light in a medium with index n is simply $v = {}^{c}/_{n}$, we get also:

$$S = ct, \tag{3}$$

so the optical length S is just the physical distance that the light would have traveled if there was no material to cross.

We start with **Fermat's principle**, a powerful tool of geometric optics. This is exactly the same as the *minimum action principle* in mechanics, and follows a similar derivation. With its aid we can solve complicated problems in a straightforward way, at the expense of a more formal approach. The Fermat principle states that:

Fermat Principle

> The path followed by a light ray is always such that its optical length S is an extreme: either a local minimum, a maximum or an inflection point.

The optical path of a ray passing through a non-homogeneous medium along a curved path γ can be calculated as

$$S = \int_{\gamma} n(\boldsymbol{x})\, d\ell \tag{4}$$

The scalar field $n(\boldsymbol{x}) = n(x, y, z)$ gives the refractive index in space as the light moves by a $d\ell$ segment. If we parametrize the curve γ with the time t the light takes to move over it, we reach:

$$d\ell = vdt = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}dt, \tag{5}$$

where we used the fact that the total speed v of the light is just the quadratic sum of the different components of the velocity. We will write also $\boldsymbol{v} = \dot{\boldsymbol{x}}$ (as vectors) and $v = |\dot{\boldsymbol{x}}| = {}^{d\ell}/_{dt}$ (scalar). So in the end the extremization problem of the optical path corresponds at taking the *functional derivative* on the possible curves γ and setting it to zero:

$$\delta \left(\int_{\gamma} n(\boldsymbol{x}) \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} dt \right) = 0 \tag{6}$$

This type of problem is solved by the **Euler-Lagrange equations** for the **Lagrangian function**:

Optical Lagrangian

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = n(\boldsymbol{x})|\dot{\boldsymbol{x}}| \tag{7}$$

The Lagrangian is a function of the *phase-space coordinates* $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ of the system (position and velocity). For a mechanical system, the Lagrangian is just the kinetic energy of the system minus the potential energy. The **Euler-Lagrange equations**, which solve the above problem, are:

Euler-Lagrange Equations

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] = 0 \tag{8}$$

These are a set of partial differential equations. By solving the Euler-Lagrange equations for a given L we obtain the **equation of motion** of the system described by L.

With the optical Lagrangian, the Lagrange equations become:

$$\frac{\partial n}{\partial x}\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} - \frac{d}{dt}\left[n\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\right] = 0 \tag{9}$$

We divide by the square root term, recalling that $|\dot{\boldsymbol{x}}| = {}^{d\ell}/_{dt}$, and obtain:

$$\frac{\partial n}{\partial x} - \frac{dt}{d\ell}\frac{d}{dt}\left[n\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\right] = 0, \tag{10}$$

where we used the fact that the inverse of the total derivative ${}^{d\ell}/_{dt}$ is the derivative of the inverse ${}^{dt}/_{d\ell}$. The latter simplifies with the total derivative on t:

$$\frac{\partial n}{\partial x} - \frac{d}{d\ell}\left[n\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\right] = 0 \tag{11}$$

Finally, since:

$$\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = \frac{dx}{dt}\frac{dt}{d\ell} = \frac{dx}{d\ell}, \tag{12}$$

with a simple rearrangement we get:

$$\frac{d}{d\ell}\left[n\frac{dx}{d\ell}\right] = \frac{\partial n}{\partial x}, \tag{13}$$

and this is valid for all components, so we finally reach:

Ray Equation (Eikonal)

$$\frac{d}{d\ell}\left[n\frac{d\boldsymbol{x}}{d\ell}\right] = \boldsymbol{\nabla} n \tag{14}$$

This equation is known as the equation of the ray or as the Eikonal equation (from the Greek for "image", the same root of the word "icon"). It is the optical analogous of the equation of motion of a mechanical system. The independent variable ℓ here is a parameter that describes the motion along the path of the curve. Solving the ray equation is non trivial, and obviously depends on the chosen coordinates and on the underlying refractive index field n. Note that if n is homogeneous the equation reduces to ${}^{d^2\boldsymbol{x}}/_{d\ell^2} = 0$, that is, $\boldsymbol{x}$ grows linearly with ℓ. This is just the mathematical formulation of the principle that "*light travels in a straight line*".

***Bonus**: the same result can be derived in a tensor approach similar to the one of general relativity by assuming a metric tensor:*

$$g_{\mu\nu} = \begin{pmatrix} n^2 & 0 & 0 \\ 0 & n^2 & 0 \\ 0 & 0 & n^2 \end{pmatrix} \tag{15}$$

*This gives a **geodesic Lagrangian**:*

$$L(x, \dot{x}) = \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}, \tag{16}$$

which is exactly the same as the Optical Lagrangian. Since the coordinate time t is an affine parameter with respect to the optical length, that is $dL/dt = c$ which is a constant, to get the equations of motion we can use the equivalent squared Lagrangian:

$$L(x, \dot{x}) = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu, \tag{17}$$

which in many contexts is easier to manipulate. Moreover, this tensor approach enables to generalize the Fermat principle to anisotropic optical media such as birefringent ones ($n_1 \neq n_2 \neq n_3$), or even the ones where the optical axes change inside the medium.

Another important relation can be derived from the Lagrangian formulation. Suppose the refractive index n does not depend on a specific variable, say x. Since $n(\boldsymbol{x})$ bears the only explicit dependence on position in the Lagrangian, if $\frac{\partial n}{\partial x} = 0$ then the Lagrangian too does not depend on x. It follows from the Lagrange equations that the momentum associated with such variable $p_x = \frac{\partial L}{\partial \dot{x}}$ is conserved along the whole path. This means:

$$\frac{\partial L}{\partial \dot{x}} = n\frac{\dot{x}}{|\dot{\boldsymbol{x}}|} = C \tag{18}$$

The fraction in the previous equation is nothing but the ratio of the velocity along the x direction to the total one. As the velocity vector has always the same direction as the light propagation, such a ratio is equal to the cosine of the angle between the propagation direction and the local unit vector $\hat{\boldsymbol{x}}$ of that coordinate. This is in turn the same as the sine of the angle α between the propagation direction and the perpendicular to $\hat{\boldsymbol{x}}$ on the plane containing both $\dot{\boldsymbol{x}}$ and $\hat{\boldsymbol{x}}$:

$$n \sin\alpha = C \tag{19}$$

In particular, since this holds for any couple of points along the path, we have proven a famous result:

Snell's Law

$$n_1 \sin\alpha_1 = n_2 \sin\alpha_2 \tag{20}$$

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Radial Index Fields

Instead of solving directly the ray equation, which could produce severe headache, to calculate the properties of atmospheric refraction we will use the polar coordinates (r, ϑ) and assume that n does not depend on the angular coordinate ϑ. Furthermore, as the curve parameter ℓ we will use the radius r itself, so that $dr/d\ell = 1$. With these choices, the optical Lagrangian can be written as:

$$L(r, d\vartheta/dr) = n(r)\sqrt{1 + r^2 \left(d\vartheta/dr\right)^2} \tag{21}$$

The conservation of $p_\vartheta = C$ imposes that:

$$\frac{n(r) r^2 \frac{d\vartheta}{dr}}{\sqrt{1 + r^2 \left(\frac{d\vartheta}{dr}\right)^2}} = C \tag{22}$$

Note that this differs from Snell's law derived just before, as the derivative $\dot{\vartheta}$ is replaced here with the one with respect of the radius r. Still, we can identify the term:

$$\frac{r\frac{d\vartheta}{dr}}{\sqrt{1+r^2\left(\frac{d\vartheta}{dr}\right)^2}} = \sin\zeta, \tag{23}$$

where $\zeta(r)$ is the **local zenith angle**, the angle of the path to the local radial direction. The equation becomes then:

Impact equation

$$nr\sin\zeta = C \tag{24}$$

This is valid locally on every point of the ray. The value of C produces a family of light rays, and can be fixed by the boundary condition $\lim_{r\to\infty} n(r) = 1$: the refractive index far from the Earth is just the one of the vacuum. This gives $C = r\sin\zeta$ for large r, which is just the original **Impact Parameter** of the ray, before it gets distorted.

Some important facts follow from the impact equation. First, a radial refractive index field can only *deflect* light rays, without altering their incoming and outgoing impact parameters, as the aforementioned limit is valid both for the ingoing and outgoing limits of the light path. As a limit case, it can be verified that if n is a constant, and expressing ζ as a function of ϑ and of the closest approach angle φ, the previous relation is just the polar equation of a straight line. Second, the zenith distance at an observation point at $r = R$ can be mapped promptly in a value of the undisturbed impact parameter C and *this association depends only of the refractive index $n(R)$ at the observer radius*:

$$C(\zeta) = n(R)\cdot R\sin\zeta \tag{25}$$

In particular, the ray reaching the observer from the horizon (tangent to the circle of radius R) has an impact parameter equal to $C = nR$. Since $n \geq 1$ we conclude that the presence of the atmosphere (whatever its refractive index is!) allows to *see objects that would otherwise be beyond the horizon*[1]. However, the impact parameter C alone is not sufficient to characterize the atmospheric distortion. This is because there are infinite undisturbed rays with the same impact parameter C, but different incoming direction. To fully describe the path, one needs to integrate Formula 22 with proper boundary conditions. We do this in the next paragraph.

By changing variable to $u = {}^1\!/_r$ and with some algebraic manipulation, Equation 22 can be cast in the following form:

$$\frac{d\vartheta}{du} = -\frac{C}{\sqrt{\mathfrak{n}^2(u) - C^2u^2}}, \tag{26}$$

where $\mathfrak{n}(u) = n(\,r(u)\,) = n({}^1\!/_u)$ is the composition of $n(r)$ with $r(u)$. This can be used to obtain the following integral form, where we have set one of the integration extremes of $d\vartheta$ to the boundary condition at infinity:

Path of the Ray

$$\vartheta(r) = Z - C\int_0^{1/r} \frac{du}{\sqrt{\mathfrak{n}^2(u) - C^2u^2}} \tag{27}$$

[1]The horizontal ray may come from a direction above the horizon of the observer, but in such case its path would have to wind around the whole Earth. This is a consequence of the conservation of p_ϑ: to change its sense of rotation, the ray would have to have $\partial\vartheta/\partial r = 0$ at least in one point. However, by Equation 22, if this happens in one point than it must happen for all of them. In short, the only rays that don't get twisted are the radial ones . All the others are twisted and they can not change their whirling direction.

In the integration we made use of the fact that for $r \to \infty$ and $u \to 0$ the angle $\vartheta \to Z$, the undistorted zenith distance of the star being observed. The integration of the latter equation for a specific n field and initial conditions Z and C gives the path of the ray in polar coordinates. The value of C can be fixed by an observation of the star at any point along the path, using Equation 24. Depending on $n(r)$, it may be more useful to express the integral directly in terms of the radial distance:

$$\vartheta(r) = Z + C \int_{-\infty}^{r} \frac{d\rho}{\rho^2 \sqrt{n^2(\rho) - C^2/\rho^2}} \tag{28}$$

As we are interested at the rays converging at one given point (so that they are observable from it), we may evaluate the path equation for a specific $\vartheta = 0$ and a given radius R. Remembering that C can also be expressed as a function of R and the observed zenith distance ζ, we finally reach the equation relating the true direction of a star Z to the one it is seen at. To simplify the formula, we refer distances to the observing radius, so that $R = 1$, and call $\mathfrak{n}(1) = n_1$:

Aberration Equation

$$Z(\zeta) = n_1 \sin\zeta \cdot \int_{0}^{1} \frac{du}{\sqrt{\mathfrak{n}^2(u) - u^2 n_1^2 \sin^2\zeta}} \tag{29}$$

Inverting the above equation, that is calculating the observed zenith distance ζ given the true one Z of the star is in general not possible. However, once $\mathfrak{n}(u)$ is specified, this can be done numerically[2]. By differentiating the aberration equation with respect to ζ we can estimate the local distortion induced by the atmosphere to the image of an object. We introduce the distortion coefficient $\psi = \partial Z/\partial\zeta$. A small square which is distorted into a rectangle with the vertical side being one half of the horizontal one has $\psi = 2$. Since the integral in the above equation is on u and not ζ, the derivative can seep in the integral, and we obtain:

Distortion Coefficient

$$\psi(\zeta) = \cos\zeta \int_{0}^{1} \mathfrak{q}(u,\zeta)\Big[1 + u^2 \sin^2(\zeta) \cdot \mathfrak{q}(u,\zeta)^2\Big] du \tag{30}$$

Where we have made use of the auxiliary function $\mathfrak{q}(u,\zeta)$:

$$\mathfrak{q}(u,\zeta) = \frac{1}{\sqrt{\left[\frac{\mathfrak{n}(u)}{n_1}\right]^2 - u^2 \sin^2\zeta}} \tag{31}$$

Again, it can be shown that if $\mathfrak{n} = n_1$ everywhere, the distortion coefficient ψ is independent of ζ and equals one. Note that the fact that $\cos(\zeta) = 0$ at the horizon does not necessarily mean that $\psi(\pi/2) = 0$, as the integral may have a divergence too and compensate for it.

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[2]A possible way to obtain an analytical approximation is to expand the integrand in a Taylor series around some point, integrate the series and use the method given in `https://www.giovanniceribella.eu/fuere/?attachment_id=1278` (in Italian) to get the series expansion of the inverse. The first order approximation provided by this method is used later to calculate the chromatic aberration coefficient.

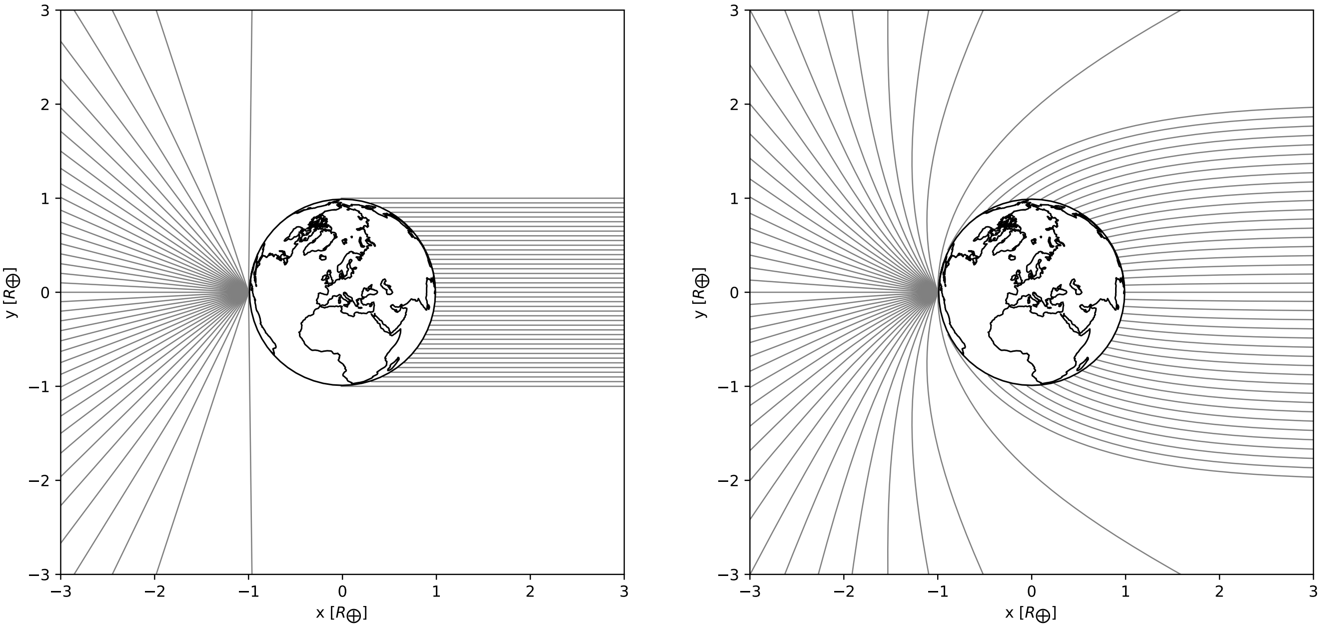

Figure 1: Paths of light rays for a realistic atmosphere (left) and a crazy one made mostly of glass (right).

Exponential Atmosphere

So far the only assumption made on $n(\boldsymbol{x})$ is that it has a polar symmetry and depends only on r. We now use a simple model of the real refractive index $n(r)$ of the terrestrial atmosphere to derive some nice plots. The refractive index of air depends, in general, on variables such as the air pressure P, its temperature T, the molar fractions of its constituents $X_{\mathrm{N_2}}, X_{\mathrm{O_2}}, X_{\mathrm{Ar}}, X_{\mathrm{CO_2}}, X_{\mathrm{H_2O}}$ and, of course, the light wavelength λ. We will treat chromatic aberrations related to λ in the next section. Fortunately, all other variables are also scalars, so that the refractive index can be given as a $n(r)$ field. Measurements found out that the profile of n with the altitude $h = r - r_0$ can be approximated by an exponential function of the kind:

Exponential Atmosphere

$$n(r) = 1 + Ae^{-m(r-r_0)} \tag{32}$$

Where A is the deviation of the refractive index at r_0 and $m > 0$ specifies the decay length. We will assume that $r_0 = R_\oplus = 6378.388\,\mathrm{km}$ and that we are observing at sea level, so $R = R_\oplus$ too. Furthermore, we will express lengths in units of $R_\oplus$. Results can be easily adapted to altitudes above the sea level just by changing the variables and shifting the index field accordingly.

Figure 1 gives the resulting ray paths for $A = 2.9 \cdot 10^{-4}$ and $m = 0.14\,\mathrm{km}^{-1}$, which are realistic parameters derived from a very rough interpolation of the atmospheric profile used for MAGIC simulations at La Palma, and for the exaggerated values of $A = 1$ and $m = 1/R_\oplus$. The left part of each drawing shows the paths of the rays converging on the leftmost point of the surface of the planet, whereas the right part presents the paths of a family of rays with equal original direction (wavefront of starlight). Rays are drawn for impact values between $-(1 + A)$ and $(1 + A)$, so the extreme rays of both families are the ones which end up horizontally at the observation site.

The effects of atmospheric refraction are barely visible for a realistic atmosphere. However we shall note that the horizontal rays at the observer site do originate from a point below his horizon: the most extreme rays in the left drawing are not originally tangent to the surface, as can be seen by their intersection point with the x axis being visibly larger than $x = -1$. The effect is hugely magnified in the "glass atmosphere" of the right plot. On such a planet, at any time roughly 85% of the sky would be visible from any point of the surface, with the distortion at the horizon being extreme. The "night" on such a world would be a rare phenomenon,

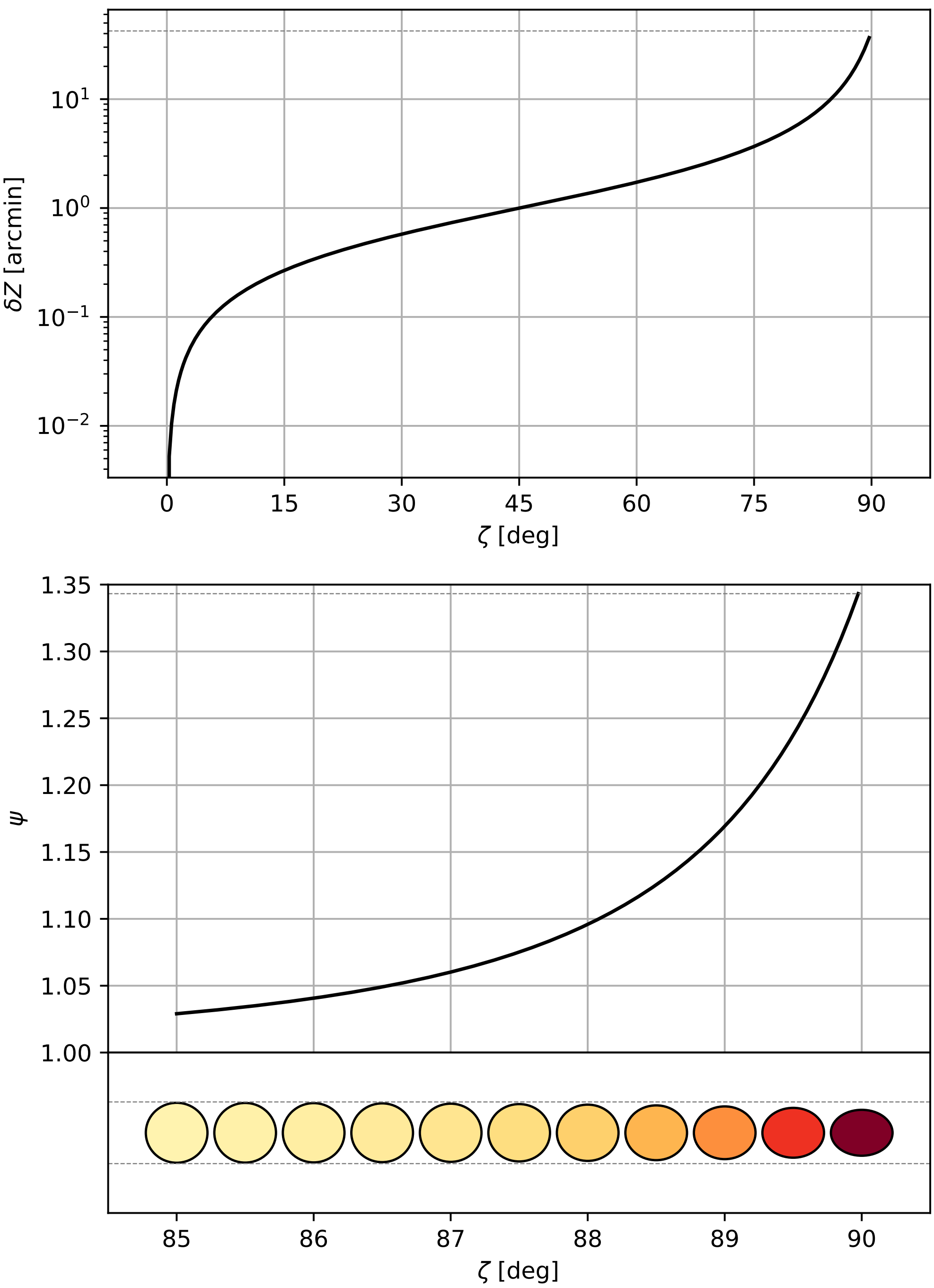

Figure 2: Plots of the positional aberration $\delta Z = Z(\zeta) - \zeta$ and of the linear distortion coefficient ψ, as a function of the observing zenith distance ζ. The Sun is affected by such a distortion as it rises or sets. The small suns in the lower plot show this effect: their color has been chosen to be proportional to ψ (and also to resemble a real sunset). The realistic values of A and m were used.

as the Sun would still be visible as a dazzling distorted rainbow even after it has set.

The aberrations induced by the terrestrial atmosphere are presented in Figure 2. The positional shift $\delta Z = Z(\zeta) - \zeta$ is zero at the zenith and increases rapidly close to the horizon, where it reaches a maximum of 42 arcmin. Since the Sun is roughly half a degree wide, this means that in a sunset over a flat and clean horizon the Sun is still visible when it is fully below it. Its apparent velocity decreases as it approaches the horizon. Moreover, its disk is distorted in the shape of an ellipse: the distortion coefficient is $\psi < 1.05$ for $\zeta < 87$ deg and then rapidly reaches the value of 1.35 at $\zeta = 90$ deg.

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Chromatic Aberrations

To treat the effects of the atmospheric dispersion, we need to introduce a dependency on the wavelength λ in the refractive index field $n(r)$. As before, we neglect the effects of absorption and diffusion. Introducing another variable can make things very complicated: we will operate a simplification and assume that the refractive index can be written as:

$$\mathfrak{n}(u, \lambda) = 1 + A(\lambda) f(u) \tag{33}$$

This is the case for the exponential atmosphere if the dispersion relation $n(\lambda)$ does not change with the altitude, that is, if only the term A in Equation 32 is a function of the wavelength, and $f(u) = \exp[-m(1/u - 1)]$ is not. The aberration equation can be derived on λ and, after some algebra, its derivative results:

$$\frac{\partial Z}{\partial \lambda}(\zeta, \lambda) = -\frac{A'(\lambda)}{n_1^3(\lambda)} \int_0^1 du\, \mathfrak{q}^3(u, \zeta, \lambda)\, \mathfrak{n}(u, \lambda) \left[f(u) - f(1) \right], \tag{34}$$

where we have indicated $A'(\lambda) = {}^{dA}/_{d\lambda}$ as a function of λ. However, ${}^{\partial Z}/_{\partial \lambda}$ is not the quantity that we search to quantify the chromatic aberration, as it indicates how much the true zenith distance Z changes with the wavelength λ for a point observed at a fixed zenith distance ζ. What we really need is the other way around, that is how ζ changes for a point at fixed Z (a star) as a function of λ. This requires us to invert the aberration equation and consider on top of that the additional dependence on λ. Fortunately, if $\psi(\zeta, \lambda) = {}^{\partial Z}/_{\partial \zeta}(\zeta, \lambda)$ is different from zero, which it is in the whole interval, we can obtain a linear approximation of both the dependence on λ and the inverse function $\zeta(Z)$ from the Taylor expansion of $Z(\zeta, \lambda)$:

$$\begin{aligned}
Z(\zeta, \lambda) &= Z(\zeta_0, \lambda_0) + \frac{\partial Z}{\partial \zeta}\bigg|_{\substack{\lambda_0\\ \zeta_0}} (\zeta - \zeta_0) + \frac{\partial Z}{\partial \lambda}\bigg|_{\substack{\lambda_0\\ \zeta_0}} (\lambda - \lambda_0) + \\
&+ \frac{1}{2}\frac{\partial^2 Z}{\partial \zeta^2}\bigg|_{\substack{\lambda_0\\ \zeta_0}} (\zeta - \zeta_0)^2 + \frac{\partial^2 Z}{\partial \zeta \partial \lambda}\bigg|_{\substack{\lambda_0\\ \zeta_0}} (\zeta - \zeta_0)(\lambda - \lambda_0) + \frac{1}{2}\frac{\partial^2 Z}{\partial \lambda^2}\bigg|_{\substack{\lambda_0\\ \zeta_0}} (\lambda - \lambda_0)^2 + \ldots \\
&= \left[Z(\zeta_0, \lambda_0) + \frac{\partial Z}{\partial \lambda}\bigg|_{\substack{\lambda_0\\ \zeta_0}} (\lambda - \lambda_0) + \frac{1}{2}\frac{\partial^2 Z}{\partial \lambda^2}\bigg|_{\substack{\lambda_0\\ \zeta_0}} (\lambda - \lambda_0)^2 + \ldots \right] + \\
&+ \left[\frac{\partial Z}{\partial \zeta}\bigg|_{\substack{\lambda_0\\ \zeta_0}} + \frac{\partial^2 Z}{\partial \zeta \partial \lambda}\bigg|_{\substack{\lambda_0\\ \zeta_0}} (\lambda - \lambda_0) + \ldots \right] (\zeta - \zeta_0) + \ldots
\end{aligned} \tag{35}$$

Where in the second equality we have grouped together explicitly the terms with the same order in $(\zeta - \zeta_0)$. The zeroth order term is just $Z_0(\lambda)$, whereas the linear term is a Taylor expansion in powers of λ for ${}^{\partial Z}/_{\partial \zeta}(\zeta_0, \lambda) = \psi(\zeta_0, \lambda)$, the distortion coefficient. The leading term in such an expansion is just the value of ψ calculated for λ_0, and we will neglect higher order terms in it. Thus, the equation can be linearized in ζ as:

$$[Z(\zeta, \lambda) - Z_0(\lambda)] = \psi(\zeta_0, \lambda_0)(\zeta - \zeta_0) \tag{36}$$

This relation can be inverted for $\delta\zeta = \zeta - \zeta_0$. Furthermore, expanding again $Z_0(\lambda)$ and discarding further dependencies on $\delta Z = Z - Z_0(\lambda_0)$ we reach:

$$\delta\zeta = -\frac{\frac{\partial Z}{\partial \lambda}\Big|_{\substack{\lambda_0\\ \zeta_0}}}{\psi(\zeta_0, \lambda_0)} \delta\lambda \tag{37}$$

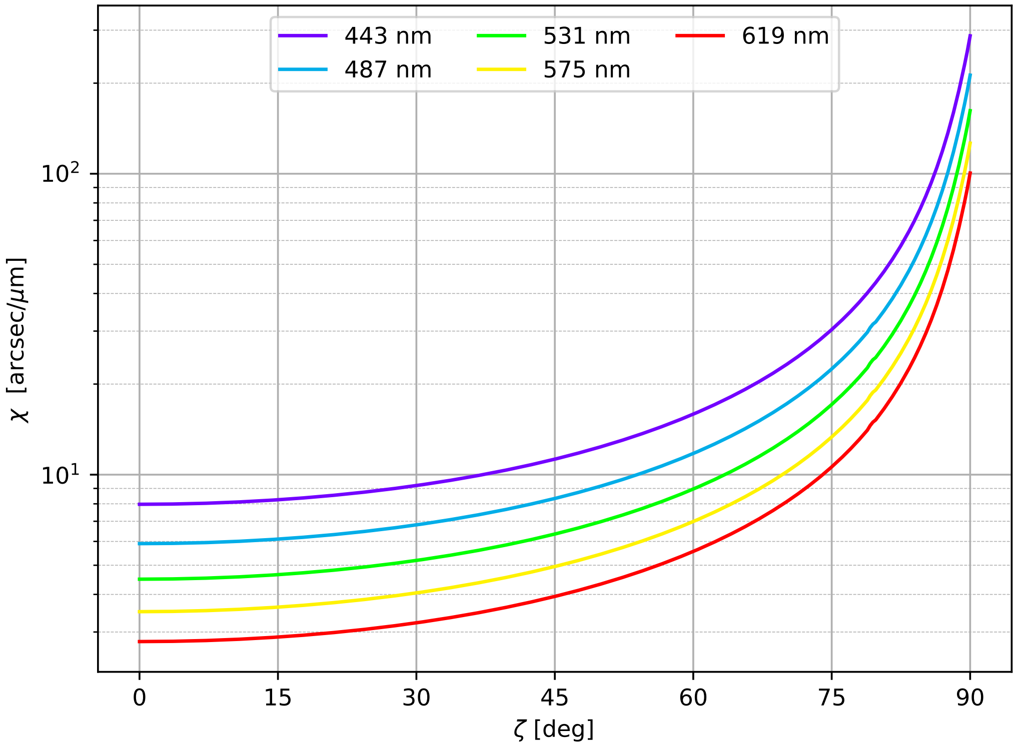

Figure 3: Linear chromatic aberration χ coefficient as a function of ζ. The value is given in $^{\text{arcsec}}/_{\mu\text{m}}$. The span of the optical spectrum is $\Delta\lambda \simeq 0.3\,\mu$m. As a result, at a given ζ we expect the size of the aberrated image to be of the order of $\chi\Delta\lambda$.

The linear chromatic aberration coefficient $\chi(\zeta, \lambda)$ can thus be estimated as:

Chromatic Aberration Coefficient

$$\chi(\zeta,\lambda) = \frac{A'(\lambda)}{n_1^3(\lambda)\psi(\zeta,\lambda)} \int_0^1 du\, \mathfrak{q}^3(u,\zeta,\lambda)\, n(u,\lambda) \left[f(u) - f(1)\right] \tag{38}$$

In order to calculate $\chi(\zeta, \lambda)$ we need to know the dispersion relation $A(\lambda)$ for air. This was derived from the work of P. Ciddor[3] and can be written as:

$$A(\lambda) = \frac{0.05792105}{238.0185 - \lambda^{-2}} + \frac{0.00167917}{57.362 - \lambda^{-2}}, \tag{39}$$

with λ expressed in μm. Figure 3 gives the trend of the linear chromatic coefficient χ as a function of the observed zenith distance ζ. Such coefficient is not dimensionless, as ψ and is measured in arc-seconds per unit wavelength of the incident light. The dispersion is stronger for shorter wavelengths, with blue and violet being the most dispersed ones. Referring to a green light of $\lambda_G = 531\,$nm, at the horizon we find $\chi_G = 150\,^{\text{arcsec}}/_{\mu\text{m}}$. Since the visible spectrum extends roughly from 400 nm to 700 nm, the atmospheric dispersion causes the light of a star grazing the horizon to be spread over $\delta_G = \chi_G \delta\lambda \simeq 0.75\,$arcmin!

In Figure 4 we plot the dispersion profiles with respect to the green component $\delta_G(\zeta, \lambda)$ as a function of ζ. The profile enlarges as the star reaches the horizon, with much of the effect becoming visible in the last 15 deg of zenith distance. We note that, since the profile is calculated in zenith distance, negative δ_G mean that the ray is deflected upwards. Green, blue and violet light reach the observer from a higher altitude than yellow, orange and red one. In pristine weather conditions, this can cause brief and spectacular "flashes" of colorful light when an object (typically the Sun) is about to set. In the next page we present a reconstruction of the chromatically aberrated and distorted disk of the Sun as it sets over an imaginary horizon. The

[3] P. E. Ciddor, *Refractive index of air: new equations for the visible and near infrared*, DOI: 10.1364/AO.35.001566

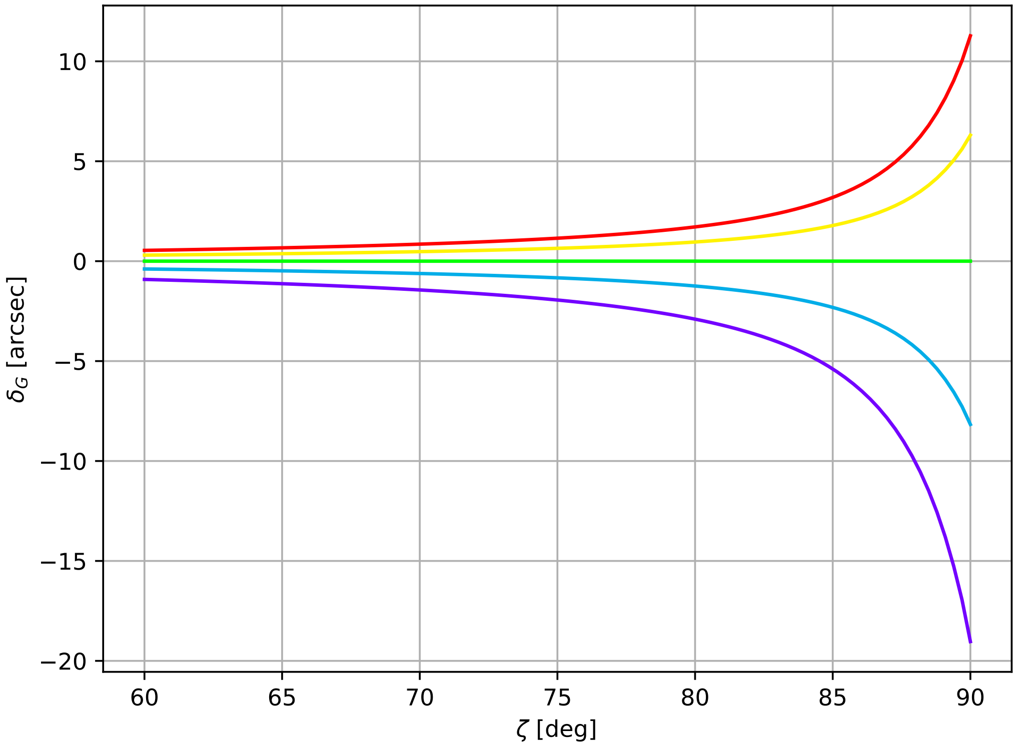

Figure 4: Chromatic aberration profiles δ_G of a point-like object viewed at zenith distance ζ, referred to its green component at 531 nm.

effects of chromatic aberration are important also for the observations of the planets, where, given the high magnification, even subtle changes can sensibly reduce the resolution and contrast of the observations.

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Further Reading

This is a nice book introducing the use of Lagrangians in optics: *Lakshminarayanan, Vasudevan & Ghatak, Ajoy & Thyagarajan, K.. (2002). Lagrangian Optics. Springer Verlag, DOI: 10.1007/978-1-4615-1711-5*. It does not stress too much the formalism and gives the important results, including an example of the atmospheric aberrations induced by a planar atmosphere (mirages in the desert).

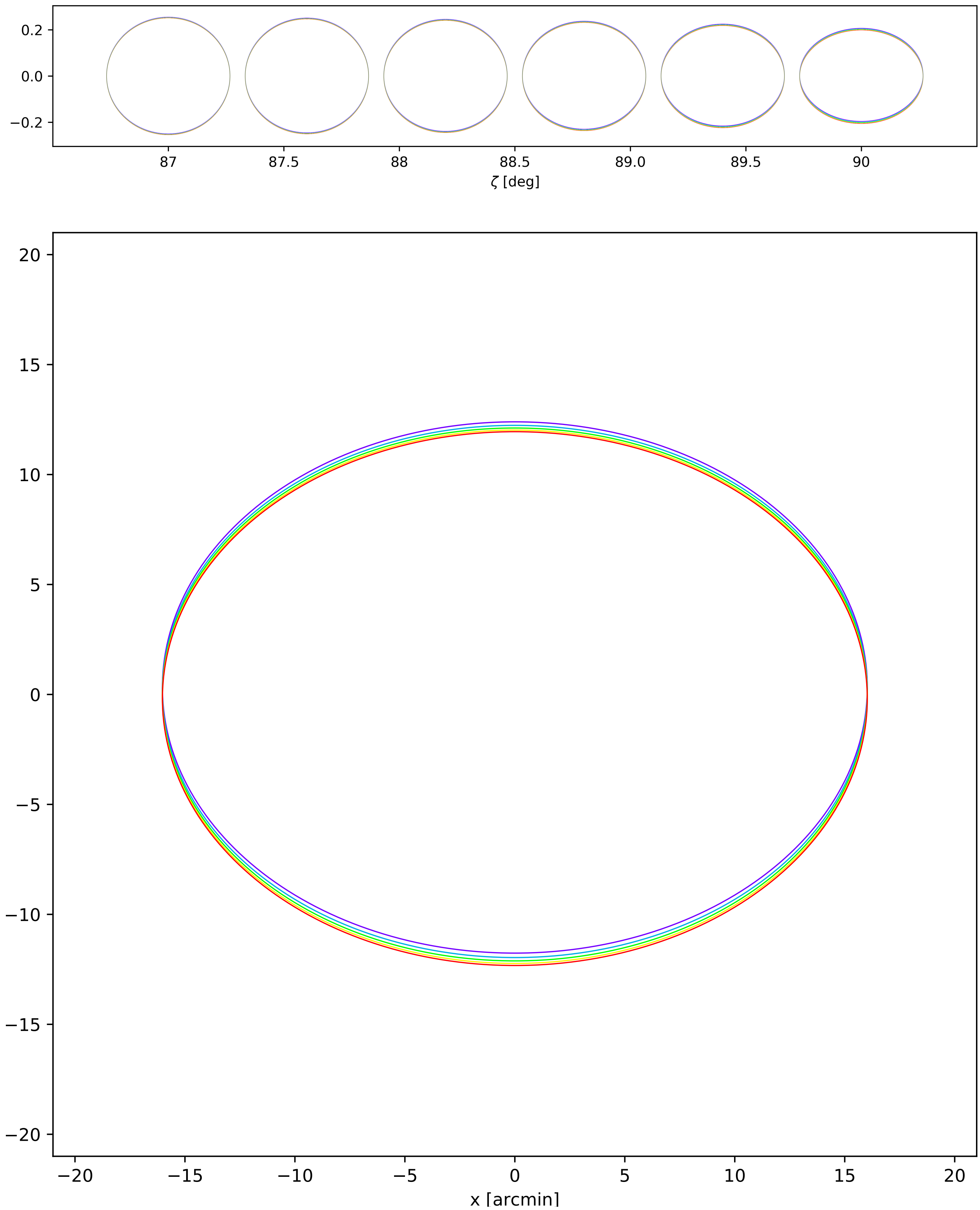

Figure 5: Composite chromatic and geometric aberrations for the Sun. Top panel: images of the Sun viewed at zenith distances close to the horizon for the same frequencies as in Figure 3. Bottom panel: zoom on the image of Sun at the horizon. With a very terse weather, the aberration becomes visible after the Sun has set as the notorious green (or rarely blue) flash.

Addendum:
Planar Motion

I said that solving the ray equation is cumbersome. However, we will use it now to prove that in a radial field $n(r)$ the path of the rays are planar, and thus the use of two dimensional polar coordinates in Equation 21 is justified. This can also be guessed directly from the spherical symmetry possessed by the system. We note first that the term ${}^{d\boldsymbol{x}}/_{d\ell}$ in the ray equation is nothing but a unit vector parallel to the instantaneous direction of the motion. This follows from the fact that ${}^{d\boldsymbol{x}}/_{dt} = \dot{\boldsymbol{x}}$ and ${}^{d\ell}/_{dt} = |\dot{\boldsymbol{x}}| = v$, the speed of the motion. We call such a unit vector $\hat{\boldsymbol{v}}$ and rewrite the ray equation as:

$$\frac{d}{d\ell}(n\hat{\boldsymbol{v}}) = \boldsymbol{\nabla} n \tag{40}$$

Since $n = n(r)$, the gradient $\boldsymbol{\nabla} n$ is necessarily parallel to the radial direction, that is $\boldsymbol{\nabla} n = (\nabla n)\hat{\boldsymbol{r}}$, where $\hat{\boldsymbol{r}}$ is a unit vector in the radial direction. If we now multiply by vector (cross) product both sides of the previous equation by such a radial unit vector, we obtain:

$$\hat{\boldsymbol{r}} \times \frac{d}{d\ell}(n\hat{\boldsymbol{v}}) = 0, \tag{41}$$

where we have made use of the fact that the cross product of two parallel vectors is zero. We keep in mind the result of the previous equation and study now the vector quantity:

$$\mathfrak{N} = n\hat{\boldsymbol{r}} \times \hat{\boldsymbol{v}} \tag{42}$$

Since $\hat{\boldsymbol{r}} = {}^{\boldsymbol{r}}/_{r}$ and $\hat{\boldsymbol{v}} = {}^{\boldsymbol{v}}/_{v}$, we can write also:

$$\mathfrak{N} = n\frac{\boldsymbol{r} \times \boldsymbol{v}}{rv} = n\frac{\boldsymbol{\omega}}{\omega} = n\hat{\boldsymbol{\omega}}, \tag{43}$$

where we have used the definition of the *angular velocity* $\boldsymbol{\omega} = {}^{1}/_{r^2}(\boldsymbol{r} \times \boldsymbol{v})$ with respect to the pole of the radially symmetric index field. Thus, $\mathfrak{N}(t)$ is nothing but a vector with the same direction as the instantaneous angular velocity of the motion and the modulo equal to the refractive index in the particular point of the space the ray is at time t. Its time derivative is:

$$\frac{d\mathfrak{N}}{dt} = \frac{dn}{dt}\hat{\boldsymbol{\omega}} + n\frac{d\hat{\boldsymbol{\omega}}}{dt} \tag{44}$$

The derivative of a unit vector is always perpendicular to the unit vector itself. The time derivative of $\mathfrak{N}$ has therefore one component which is parallel to the angular velocity and a second one orthogonal to it. However, we have a second way to derive the time derivative of $\mathfrak{N}$, namely, by deriving it in ℓ:

$$\frac{d\mathfrak{N}}{dt} = \frac{d\ell}{dt}\frac{d\mathfrak{N}}{d\ell} = v\frac{d}{d\ell}(\hat{\boldsymbol{r}} \times n\hat{\boldsymbol{v}}) \tag{45}$$

The last equation unscrambles in

$$\frac{d}{d\ell}(\hat{\boldsymbol{r}} \times n\hat{\boldsymbol{v}}) = \frac{d\hat{\boldsymbol{r}}}{d\ell} \times n\hat{\boldsymbol{v}} + \cancel{\hat{\boldsymbol{r}} \times \frac{d}{d\ell}(n\hat{\boldsymbol{v}})} \tag{46}$$

However the second term of the addition is zero by Equation 41! We have so:

$$\frac{d\mathfrak{N}}{dt} = v\frac{d\hat{\boldsymbol{r}}}{d\ell} \times n\hat{\boldsymbol{v}} = \frac{d\hat{\boldsymbol{r}}}{dt} \times n\hat{\boldsymbol{v}} \tag{47}$$

The time derivative of the unit vector $\hat{\boldsymbol{r}}$ is defined by:

$$\frac{d\hat{\boldsymbol{r}}}{dt} = \boldsymbol{\omega} \times \hat{\boldsymbol{r}} \tag{48}$$

Therefore, in the end we get:

$$\frac{d\mathfrak{N}}{dt} = n(\boldsymbol{\omega} \times \hat{\boldsymbol{r}}) \times \hat{\boldsymbol{v}} \tag{49}$$

Using the triple vector product identity, this can be seen to be equal to:

$$\frac{d\mathfrak{N}}{dt} = -n(\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{r}})\boldsymbol{\omega} + \cancel{n(\hat{\boldsymbol{v}} \cdot \boldsymbol{\omega})\hat{\boldsymbol{r}}} \tag{50}$$

However, $\boldsymbol{\omega}$ is necessarily orthogonal to $\hat{\boldsymbol{v}}$ because of its own definition (the angular velocity is perpendicular both to the radius and to the instantaneous direction of the motion). Therefore the second term in the previous equation is zero. Now, by comparing directly Equation 44 and Equation 50:

$$\frac{dn}{dt}\hat{\boldsymbol{\omega}} + n\frac{d\hat{\boldsymbol{\omega}}}{dt} = -n(\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{r}})\boldsymbol{\omega}, \tag{51}$$

we see that the right hand side is parallel to $\boldsymbol{\omega}$, whereas on the left hand side a term perpendicular to it exists too. This can only happen if such a term is identically zero for each possible value of its constituting variables. This forces:

$$\frac{d\hat{\boldsymbol{\omega}}}{dt} = 0 \tag{52}$$

That is, the direction of the angular velocity of the motion is always constant, which means that the motion happens on a plane.

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